

# Electromagnetic Response of an Earth Having a Continuous Conductivity Variation in Depth

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## ABSTRACT

A new solution for the electromagnetic response of a horizontally stratified earth having an arbitrary conductivity-depth function is presented. The primary field is assumed to be a magnetic dipole. Based on a classical expression for a distinctly layered earth, a new kernel function that represents continuous changes in conductivity is derived. It is shown that the new kernel function satisfies a Riccati type differential equation for which no general solution is known to exist and, therefore, its analytic inversion is still an outstanding problem.

When reduced to the distinct layer cases, the new kernel function shows numerically that its shape and behavior are identical to those derived from the old layered model expression. Several examples computed from both the old and the new kernels are compared. The new representation is suitable for forward problems and provides a theoretical basis that can convert a wideband electromagnetic response into a continuous conductivity-depth section. The ability to handle an earth model having continuously changing conductivity is urgent and timely, particularly with the advent of broadband sensors that can collect the entire spectral response.

## Introduction

Most electromagnetic (EM) modeling of a layered earth assumes a finite number of layers, each having a distinct electrical conductivity. Such a layering is often unrealistic for gradual changes in real earth. This article presents expressions for calculating the EM response of a laterally homogeneous (i.e., 1-D) earth with a continuous conductivity variation in depth.

A practical motive for this formulation is the recent progress in broadband, multifrequency, EM sensors that can measure the earth response at many frequencies, or even the entire spectral response over a broad bandwidth (Won *et al.*, 1996, 1997). Such sensors, in principle, can generate enough data for "frequency sounding," a depth-sounding method, by changing the transmitter frequency. Won (1980, 1983) also discussed some theoretical and experimental results employing broadband and sweep-frequency EM fields. The ultimate goal is to convert the broadband spectral EM data to a continuous conductivity-depth section, similar to the velocity-depth section derived from CDP seismic data. To enhance lateral resolution, such a section may undergo a depth migration process, again similar to seismic data. The formulation presented in this article may be considered as a first step toward that goal.

## Layered Earth Model

Fundamental equations for the magnetic field generated by a vertical dipole located at the surface of a horizontally stratified earth have been given by Kozulin (1963) and Frischknecht (1967). Figure 1 depicts the geometry of

this classical problem. The vertical component  $F_z$  of the vector potential generated by a vertical dipole at a point at or above the surface of the earth is, in the notation of Koefoed *et al.* (1972), given by

$$F_z = \frac{M}{4\pi} \left( \frac{1}{\sqrt{r^2 + (h_T - h_R)^2}} + \int_0^\infty R(\lambda, d_m, \sigma_m, \omega) e^{-\lambda(h_T + h_R)} J_0(\lambda r) d\lambda \right). \quad (1a)$$

The components of the vector potential in the horizontal plane are zero, and the displacement current in the earth is ignored.

We consider the case when both the transmitter and receiver loops are horizontal with respect to the earth. The mathematics presented here, however, is applicable to other configurations such as vertical coplanar, vertical coaxial, or perpendicular loops [for definitions and mathematical derivations, see, for example, Frischknecht (1967), p. 339]. As shown in Fig. 1, the 1-D earth is modeled by a finite number of horizontal layers, each having a distinct electrical conductivity and thickness. The last (deepest) layer has an infinite thickness. In the above equation, the vertical coordinate  $z$  increases downward, with  $z = 0$  on the surface. The notations used are defined as follows:

- $d_m$  thickness of the  $m$ -th subsurface layer,
- $\sigma_m$  conductivity of the  $m$ -th subsurface layer,
- $h_T$  height of the transmitter above the ground,
- $h_R$  height of the receiver above the ground,
- $r$  horizontal distance between transmitter and receiver,
- $M$  dipole moment of the transmitter,

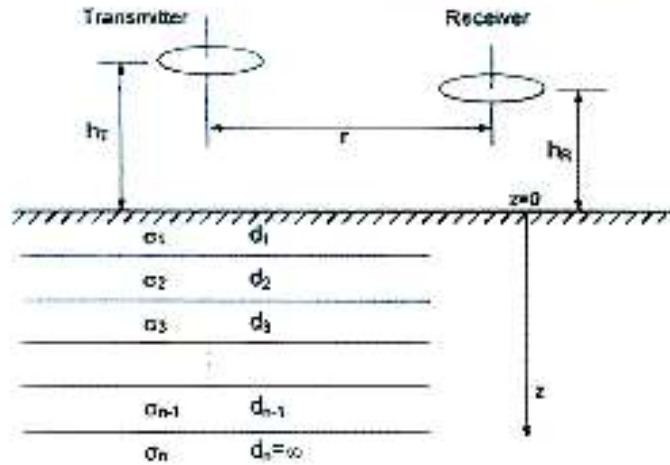


Figure 1. Geometry of the sensor and layered earth.

$\omega$  angular frequency of the transmitted signal, and  $J_0$  0-th order Bessel function.

The kernel  $R(\lambda)$  in Eq. (1a) can be computed using a recurrence relation given by Koefoed *et al.* (1972). Setting  $R(\lambda) = R_{0,n}(\lambda)$  in which the first suffix 0 represents the field in space above the ground surface, and the second suffix  $n$  represents the number of subsurface layers, we have

$$R_{n-1,n} = \frac{V_{n-1,n} + R_{n,n}(\lambda)e^{-2d_n V_n}}{1 + V_{n-1,n}R_{n,n}(\lambda)e^{-2d_n V_n}} \quad (1b)$$

with

$$R_{n,n}(\lambda) = 0 \quad (1c)$$

where

$$V_n = \sqrt{\lambda^2 + k_n^2} \quad (1d)$$

$$k_n^2 = i\omega\mu_0\sigma_n \quad \text{and} \quad (1e)$$

$$V_{n+1} = \frac{V_n - V_n}{V_n + V_n} \quad (1f)$$

The relation between the magnetic field strength  $H$  and the vector potential  $F$  is

$$H = \nabla\nabla \cdot F - k^2 F \quad (2a)$$

which, in the space above the ground surface, where  $k^2 = k_0^2 = i\omega\mu_0\sigma_0 = 0$ , reduces to

$$H = \nabla\nabla \cdot F \quad \text{or} \quad (2b)$$

$$H_z = \frac{\partial^2 F_z}{\partial z^2} \quad \text{and} \quad H_r = \frac{\partial^2 F_z}{\partial z \partial r} \quad (2c)$$

Applying the differentiation to equation (1a) gives

$$H_z = -\frac{M}{4\pi r'^3} \left[ 1 - 3 \left( \frac{h_T - h_R}{r'} \right)^2 \right] + \frac{M}{4\pi} \int_0^\infty \lambda^2 R(\lambda, d_n, \sigma_n, \omega) e^{-\lambda(h_T + h_R)} J_0(\lambda r) d\lambda \quad (3a)$$

with

$$r'^2 = r^2 + (h_T - h_R)^2 \quad \text{and}$$

$$H_r = \frac{M}{4\pi} \int_0^\infty \lambda^2 R(\lambda, d_n, \sigma_n, \omega) e^{-\lambda(h_T + h_R)} J_1(\lambda r) d\lambda \quad (3b)$$

where  $J_0$  and  $J_1$  are Bessel functions of the first kind. The first term in equation (3a) represents the primary field in the absence of the earth, while the second term denotes the secondary field. Only the secondary field exists for the radial component expressed by (3b).

The recurrence formula (1b) is widely used for computing the EM responses of multi-layered earth. The integrals (3a) and (3b) are often computed numerically using a Hankel transform algorithm or a digital filter method (Koefoed *et al.*, 1972; Anderson, 1979, 1982; Chave, 1983).

### Continuous Conductivity Model

As a first approach, consider a slowly changing conductivity function  $\sigma(z)$ , *i.e.*, no abrupt changes across layer boundaries. We transform the kernel function  $R(\lambda)$  into a form that would allow a continuously varying conductivity function  $\sigma(z)$ . Our goal is to express the magnetic field as an integral equation containing a continuous function  $\sigma(z)$ . In order to change  $R(\lambda)$  to a differential form, we first let the thickness of each layer  $d_n$  be  $\Delta z$ , an infinitesimal distance. Under this condition, the terms appearing in (1b) can be substituted as follows:

$$V_{n,n-1} = \frac{V_{n-1} - V_n}{V_{n-1} + V_n} = \frac{-1}{4(\lambda^2 + k^2)} \frac{dk^2}{dz} \Delta z \quad \text{and} \quad (4a)$$

$$d_n V_n = \Delta z \sqrt{\lambda^2 + k^2} \quad (4b)$$

Here we assume that  $k^2(z)$  [proportional to  $\sigma(z)$ ] is piecewise continuous and differentiable. Thus, the conditions exclude the traditional layered earth model consisting of distinct layers. While the two numerator terms in equation (1b) are in the first order of  $\Delta z$ , their product appearing in the denominator is in the second order of  $\Delta z$ .

We also let  $h = z = 0$ , *i.e.*, both the transmitter and receiver loops are located on the  $z = 0$  surface. This condition does not lose generality since an air space between the loops and ground can be specified as a layer of zero conductivity. For the limiting case of vanishing  $\Delta z$ , therefore, we may write Equation (1b) as

$$R_{n-1,n}(\lambda) = V_{n-1,n} + R_{n,n}(\lambda)e^{-2d_n V_n} \quad (5)$$

Expanding this and using its own recurrence relationship, we find

$$\begin{aligned} R(\lambda) &= R_{0,n}(\lambda) \\ &= V_{0,1} + V_{1,2}e^{-2d_1 V_1} \\ &\quad + V_{2,3}e^{-2d_1(V_1 + d_2 V_2)} + \dots + V_{n-1,n}e^{-2d_1(V_1 + \dots + d_{n-1} V_{n-1})} \\ &= \sum_{n=0}^{\infty} V_{n,n+1} \exp\left(-2 \sum_{n=1}^{\infty} d_n V_n\right) \end{aligned} \quad (6)$$

As we let  $\Delta z \rightarrow 0$ , we may express (6), using (4a) and (4b), in an integral form:

$$R(\lambda) = -\frac{1}{2} \int_0^\infty \frac{1}{V} \frac{dV}{dz} \exp\left(-2 \int_0^z V(\xi) d\xi\right) dz \quad (7a)$$

where

$$V(z) = \sqrt{\lambda^2 + k^2(z)}. \quad (7b)$$

The exponential term within the integrand in (7a) implies a two-way attenuation in depth, while the remaining terms may be interpreted as an inductive coupling within the earth medium. Since the exponential term is dominant, we would expect a stable and convergent integration. The secondary vertical magnetic field can now be written as

$$H_z^i(\omega, r) = -\frac{M}{8\pi} \int_0^\infty \lambda^2 J_0(\lambda r) d\lambda \\ \times \int_0^\infty \frac{1}{V(z)} \frac{dV(z)}{dz} \exp\left(-2 \int_0^z V(\xi) d\xi\right) dz. \quad (8)$$

Changing the order of integration between  $z$  and  $\lambda$ , and using the relationship

$$\frac{1}{V} \frac{dV}{dz} = \frac{1}{2(\lambda^2 + k^2)} \frac{dk^2}{dz}, \quad (9)$$

we may rewrite (8) as

$$H_z^i(\omega, r) = -\frac{M}{8\pi} \int_0^\infty \frac{dk^2(z)}{dz} dz \\ \times \int_0^\infty \frac{\lambda^2 J_0(\lambda r)}{\lambda^2 + k^2(z)} \exp\left[-2 \int_0^z \sqrt{\lambda^2 + k^2(\xi)} d\xi\right] d\lambda. \quad (10)$$

The  $\lambda$ -integral can be evaluated by the residue theorem. We first note that the integral is symmetrical in  $\lambda$ ; thus, its integration range can be expanded to negative infinity. We find two simple poles at  $\lambda = \pm ik$ . Also, the integration along an infinite semicircle in the upper half of the complex plane can be shown to be zero. The integral, therefore, can be represented by the residue at  $\lambda = ik$ , and, thus, becomes

$$H_z^i(\omega, r) = -\frac{M}{16} \int_0^\infty k(z) \frac{dk^2(z)}{dz} J_0(ikr) \\ \times \exp\left[-i \int_0^z \sqrt{k^2(z) - k^2(\xi)} d\xi\right] dz \quad (11)$$

where  $k^2(z) = i\omega\mu_0\sigma(z)$ . It is interesting to observe in (11) that the term  $\sqrt{k^2(z) - k^2(\xi)}$ , depending on the sign of  $\sigma(z) - \sigma(\xi)$ , affects the phase of the EM field that propagates and attenuates in the  $z$ -direction. Equation (11) represents

a significant improvement over (3a) in the sense that it is the expression by which one can compute the EM response over an earth having a continuous conductivity function. The exponential attenuation term insures integration stability. The next goal would be to invert equation (11) so that  $\sigma(z)$  can be derived from a given wideband spectrum  $H(\omega, r)$ . This would be an ideal "frequency-sounding" method.

### Ricatti Representation

We now consider a second approach where we find that the kernel  $R(\lambda)$  satisfies a differential equation known as the Ricatti function. To derive the equation for  $R$ , we again start from the layered earth model and consider  $R_{m,n}$  from (1b) as the function of its first subscript  $m$ . The increase of  $m$  corresponds to the increase of the  $z$  coordinate, and, thus,  $R$  can be considered also as a function of  $z$ . Similarly,  $V_m$  can be viewed as a function of  $m$ , or as a function of the vertical coordinate  $z$ . We assume that all the layers have the same thickness,  $d_m = \Delta z$ . Let the difference in  $R_{m,n}$  due to the change of index  $m$  be  $\Delta R_{m,n} = R_{m,n} - R_{m-1,n}$ . Using equation (1b), the difference can be expressed as:

$$\Delta R_{m,n} = \frac{R_{m,n}(1 - e^{-2\Delta z V_m}) - V_{m-1,n}(1 - R_{m,n}^2 e^{-2\Delta z V_m})}{1 + V_{m-1,n} R_{m,n} e^{-2\Delta z V_m}}. \quad (12)$$

Consider now the infinitesimally small change in  $z$ , i.e., the limit  $\Delta z \rightarrow 0$ . The terms in (12) reduce to

$$1 - e^{-2\Delta z V_m} \rightarrow 2\Delta z V_m, \quad 1 - R_{m,n}^2 e^{-2\Delta z V_m} \rightarrow 1 - R_{m,n}^2,$$

and for the continuous  $V$ ,

$$V_{m-1,n} \rightarrow -\Delta V_n / 2V_n.$$

We now drop the index  $m$  and consider the dependence on  $z$  instead. Changing the notation from  $\Delta R$ ,  $\Delta V$  and  $\Delta z$  to  $dR$ ,  $dV$  and  $dz$ , respectively, and disregarding  $(dz)^2$ , we get the differential equation for  $R_n(z)$ :

$$R_n'(z) = 2V(z)R_n(z) + \frac{V'(z)}{2V(z)}[1 - R_n^2(z)]. \quad (13a)$$

So far, we considered only the limit of  $\Delta z \rightarrow 0$ , which, of course, made the number of layers infinite. The total layer thickness above the homogeneous halfspace did not change, however. The subscript  $n$  of the function  $R_n(z)$  indicates that the boundary condition for (13a) should be transferred from (1c), which is now written as:

$$R_n(z)\Big|_{z=-z_{z=1, \infty}} = 0. \quad (13b)$$

If the halfspace with the properties changing up to  $z = \infty$  is considered, equations (13a) and (13b) should be rewritten as

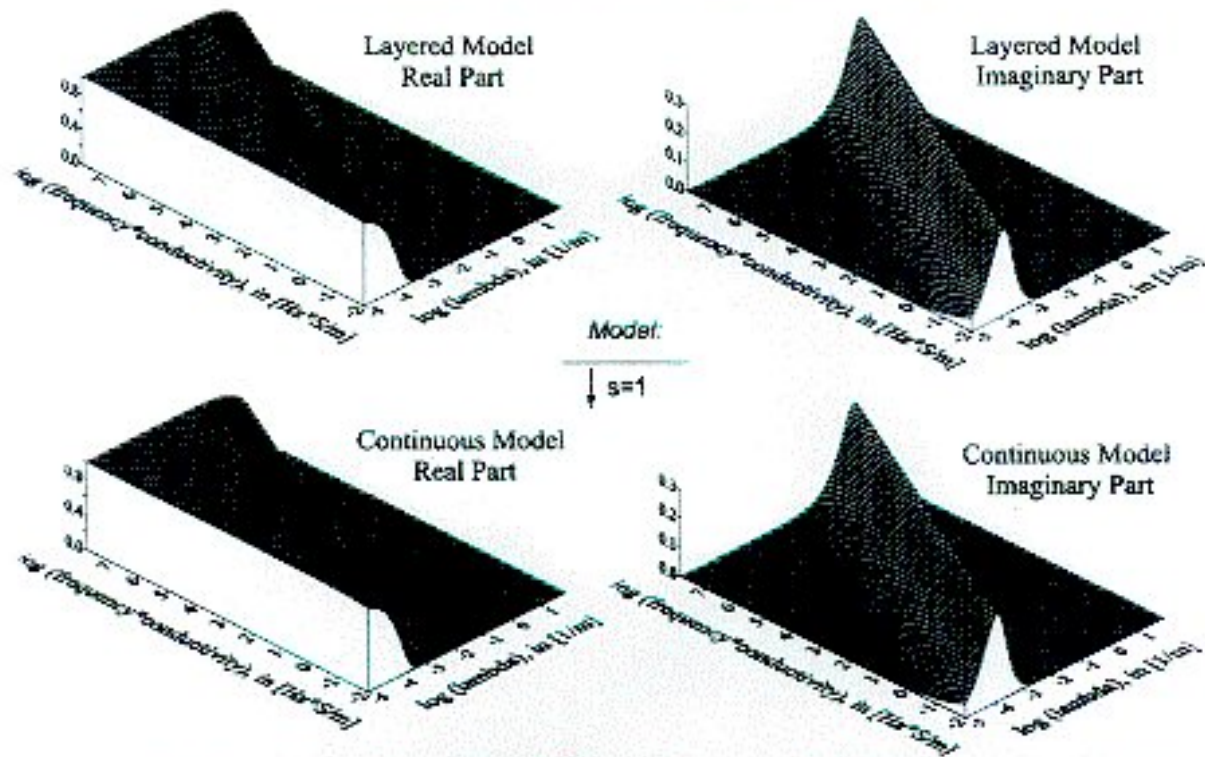


Figure 2. Comparison of two kernel functions ( $-R$ ) for a uniform conductive halfspace, computed from layered (top) and continuous (bottom) equations. The kernel is plotted as a function of the parameter lambda and the product of the frequency and the conductivity of the halfspace.

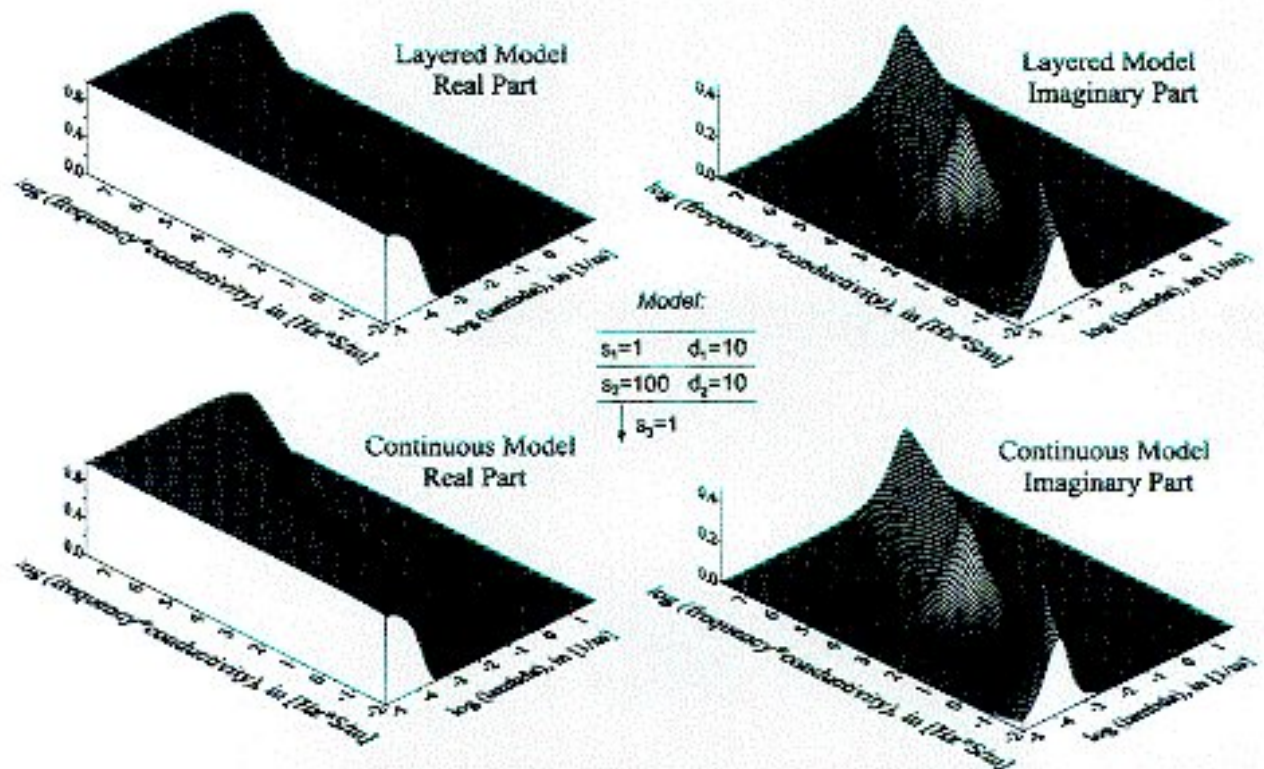


Figure 3. Comparison of two kernel functions ( $-R$ ) for a case of two layers over a uniform halfspace, computed from layered (top) and continuous (bottom) equations.

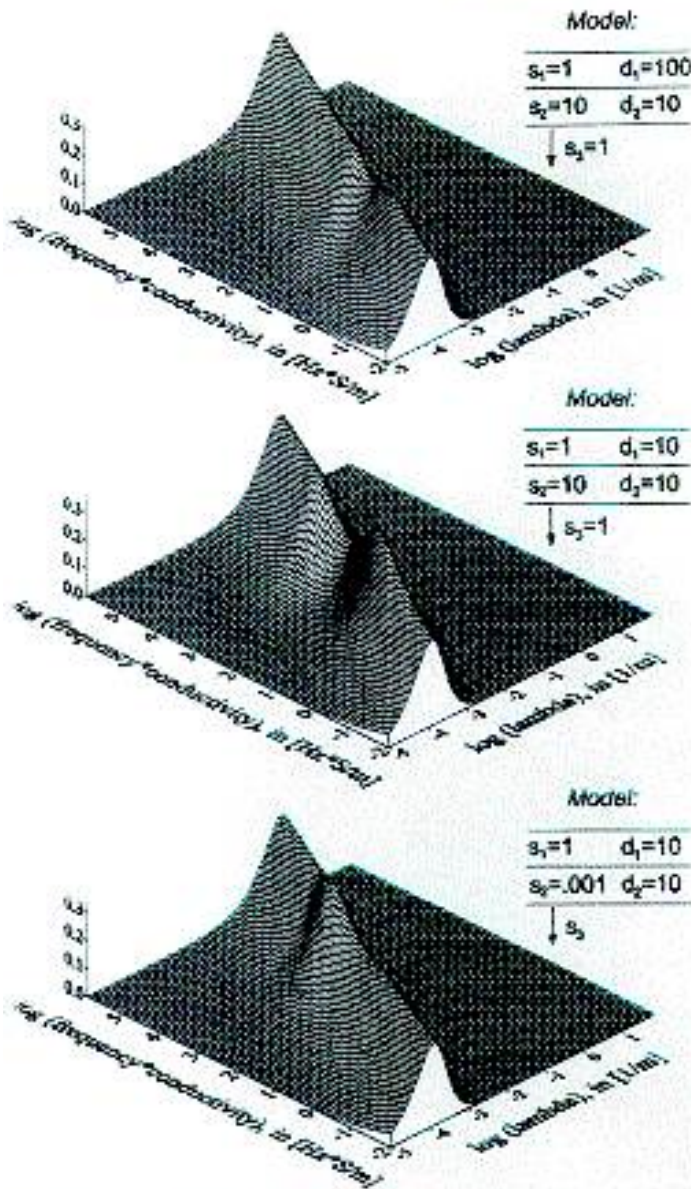


Figure 4. Kernel function  $-R$  (imaginary part only) for three heuristic examples computed from the continuous equation.

$$R'(z) = 2C(z)R(z) + \frac{V'(z)}{2V(z)}[1 - R^2(z)] \quad (14a)$$

with a boundary condition

$$R(z)|_{z \rightarrow \infty} = 0. \quad (14b)$$

The solution of (14a) at  $z = 0$  gives the kernel function for the integrals in equation (3):

$$R(z)|_{z=0} = R(\lambda, \sigma, \omega). \quad (15)$$

Equation (14a) is of the Riccati type (see, for example, Bender and Orszag, 1978). Unfortunately, no universal method for solving Riccati equations is known.

#### Weak Conductivity Contrasts

If the changes in material properties are small, and because  $R$  is a combination of the constants describing material properties, the approximation  $R \ll 1$  may be considered. Indeed, omitting a complicated proof here, we notice that, for a homogeneous halfspace, equation (1f) gives

$$R_{\text{halfspace}} = \frac{V_1 - V_0}{V_1 + V_0} \rightarrow 0 \quad \text{when } V_1 \rightarrow V_0$$

and, when  $R \ll 1$ , the Riccati equation (14a) simplifies to

$$R' - 2RV = \frac{V'}{2V} \quad (16)$$

which, with the boundary condition (14b), has a solution:

$$R(z) = -\exp\left(2 \int_0^z V dx\right) \int_z^\infty \frac{V'}{2V} \exp\left(-\int_0^x V d\xi\right) dx. \quad (17)$$

The kernel function is then:

$$R(\lambda, \sigma, \omega) = R(z)|_{z=0} = -\int_0^\infty \frac{V'}{2V} \exp\left(-\int_0^x V d\xi\right) dx. \quad (18)$$

The kernel in (18) coincides with the results obtained in the previous section as expressed by equations (8) and (11). As we see now, this function corresponds to the case of a slow change of properties of the conducting medium. Unfortunately, there is always at least one abrupt change of properties, and that is the discontinuity between the air and the earth;  $R_{0,0}$  corresponds to the field just above the earth's surface. Thus, equation (18) is applicable only to the case when the receiver is in touch with the halfspace.

#### Sharp Discontinuity

Although equation (14a) was formally derived with the assumption of continuous  $V(z)$ , there is no limit on the derivatives of  $V(z)$  and  $R(z)$ . Thus, we may study the limiting case of sharp discontinuity. Physically, a discontinuity can be represented as an area (interval on the  $z$ -axis) with large derivatives of the functions involved. When this is the case, the term  $2RV$  can be disregarded compared to the other two terms in (14a) which include derivatives. Then (14a) becomes

$$\frac{R'}{1 - R^2} = \frac{V'}{2V}, \quad (19)$$

which has a solution

$$\frac{1}{2} \ln \frac{1 + R}{1 - R} = \frac{1}{2} \ln(CV), \quad (20)$$

where  $C$  is an arbitrary constant. Thus, on both sides of the discontinuity, the combination

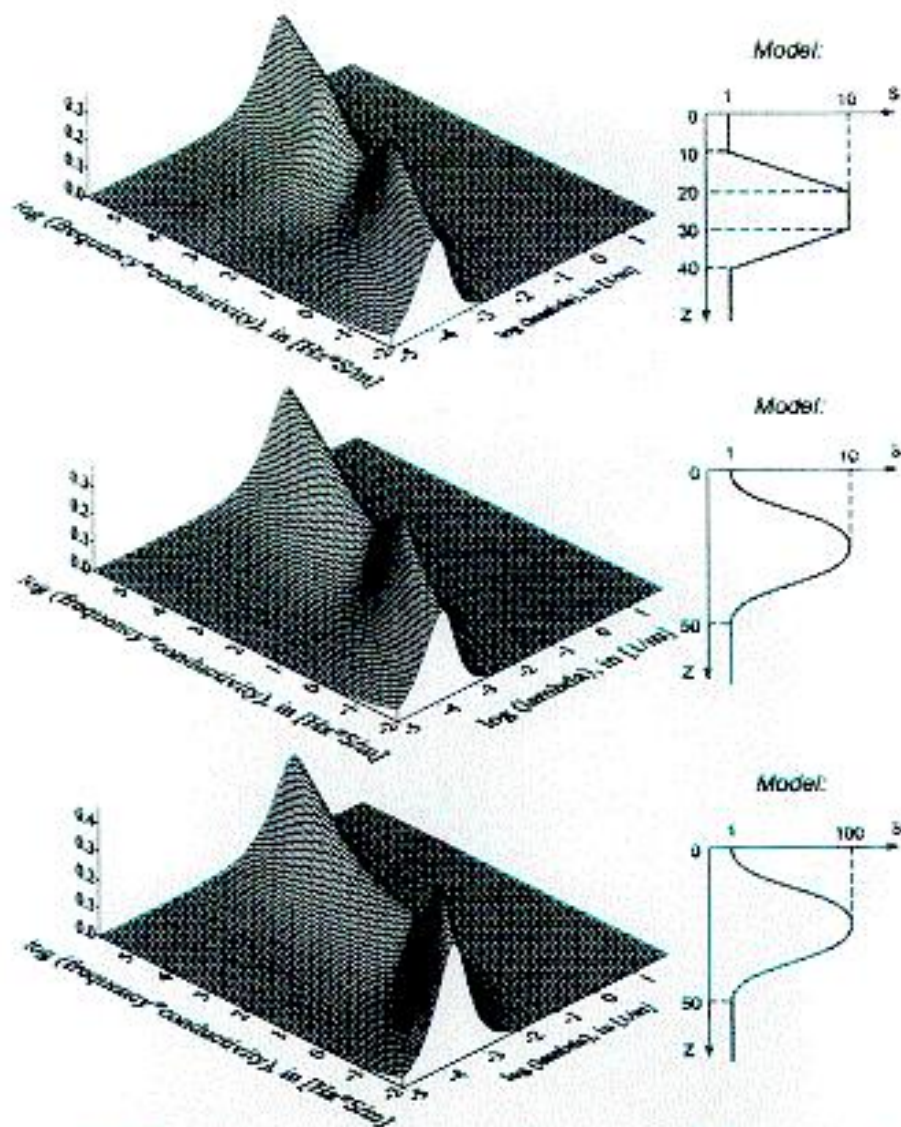


Figure 5. Kernel function  $-R$  (imaginary part only) computed from the continuous equation for three cases where resistive media bound a conductive layer.

$C = (1/V)(1+R)/(1-R)$  should remain constant. Hence, it follows that:

$$R_{m-1,n} = \frac{V_{m-1,n} + R_{m,n}}{1 + V_{m-1,n}R_{m,n}}, \quad (21)$$

which is the same as (1b), with the exception of the exponents. The exponents in (1b) reflect the decay of the field as it propagates inside a layer, while the rest of the terms describe the change in the field that occurs on the boundary between layers  $m$  and  $m-1$ . Equation (21) shows only the change across the sharp discontinuity, without accounting for the decay in the rest of the space. Thus, both equations identically describe the sharp discontinuity.

In the case that the exponential decay inside the layers does not enter into the equation (e.g., the case of a homogeneous halfspace), equations (1b) and (21) give identical results:

$$R_{0,1} = \frac{V_0 - V_1}{V_0 + V_1} = \frac{\lambda - V_1}{\lambda + V_1}. \quad (22)$$

It is interesting to note that for the free space,  $V = V_0 = \lambda$ , equation (14a) can be written as  $R' - 2\lambda R = 0$  with the solution  $R = C e^{-2\lambda z}$ , i.e., regular exponential decay.

#### Inverse Problem: Solution for $V(z)$ when $R(z)$ is Known

When the function  $R$  is known as a function of depth,  $R(z)$ , the properties of the medium,  $V(z)$ , can be obtained as follows. We first rewrite (14a) by substituting  $U = 1/V$ :

$$U' = 4R - \frac{2R'}{1 - R^2}U. \quad (23)$$

The general solution of (23) is:

$$U(z) = \exp\left(-\int_0^z \frac{2R'}{1-R^2} dx\right) \times \left[ C + 4 \int_0^z R \exp\left(\int_0^x \frac{2R'}{1-R^2} d\xi\right) dx \right] \quad (24)$$

where  $C$  is an arbitrary constant. After some manipulations and returning to the original variable  $V(z)$ , we get the solution of the inverse problem:

$$V(z) = \left\{ \frac{1-R(z)}{1+R(z)} \times \left[ \frac{1}{\lambda} \frac{1+R(0)}{1-R(0)} + 4 \int_0^z R(x) \frac{1+R(x)}{1-R(x)} dx \right] \right\}^{-1} \quad (25)$$

### Numerical Simulations

We have shown analytically that the discrete equation (1b-f) and the continuous equation (14) result in an identical expression for a uniform halfspace. In this section, we show that the numerical solutions of both equations give the same results not only for a uniform halfspace, but also for the layered models.

We compare here the kernel functions for layered and continuous cases. While the layered equation makes the solution for  $R$  readily available, the continuous equation (14) should be solved numerically. Using finite differences, we approximate equation (13a) as:

$$R_{i-1} = R_i - \left\{ 2V_i R_i + \frac{V_i - V_{i-1}}{2V_i \Delta z} (1 - R_i^2) \right\} \Delta z \quad (21a)$$

with

$$R_k = 0 \quad \text{and} \quad R_0 = R(z)|_{z=0} \quad (21b)$$

Notice that this representation is different from the one in equation (1b-f), the *true* layered equation. The thickness and the number of layers in equation (21a) are defined by the requirement of the finite difference scheme (its approximation and convergence). The regions of sharp discontinuity can be represented as a stack of thin layers with constant conductivities. The conductivity of the layers in the stack changes with the layer number  $k$ , from the value at one side of the sharp discontinuity, to the value on the other side of the sharp discontinuity. By making the stack of layers thinner, any sharp discontinuities can be reasonably represented.

In the following figures, we plot the real and imaginary parts of the kernel function  $R$  as a function of parameter  $\lambda$  and the product of the frequency  $f$  and conductivity  $\sigma$ . We use logarithmic axes for both  $\lambda$  and  $f\sigma$ . For visual ease, we choose to plot  $-R(\lambda, f\sigma)$  instead of  $R(\lambda, f\sigma)$ . As we start with a halfspace and move on to layered and con-

tinuous cases, we use the following conventions. The conductivity, as a function of layer number  $m$ , or of depth  $z$ , is described as  $\sigma_m = s_m \sigma$ , or  $\sigma(z) = s(z) \sigma$ , where  $\sigma$  is the constant conductivity of the halfspace underlying all layers, or the region with continuously changing conductivity. Thus, each model is described through the conductivity parameter  $s$ . For the underlying halfspace, the conductivity parameter is equal to one.

The solution for the uniform halfspace is shown in Fig. 2, for both the layered equation (top) and the continuous equation (bottom). The numerical representations for the two equations are identical.

Figure 3 shows the kernels of both equations in the case of a highly conductive layer buried under a layer with the same conductivity as in the underlying halfspace. We again see the remarkable agreement between the solutions obtained from the layered (top) and the continuous equations (bottom).

Figure 4 shows three heuristic examples (imaginary parts only, for simplicity) computed from the continuous equation (21a) only. The effects of thickness and conductivity contrast for various layers are studied. The results indicate that some of the parameters may be strongly coupled. For instance, the product of layer thickness and conductivity contrast may not be easily resolved, since the contribution of a thin but high conductivity layer is similar to that of a thick but low conductivity layer.

Lastly, Fig. 5 shows three cases of continuously changing conductivity in depth. These solutions, obviously, can only be obtained using the continuous equations.

### Conclusions

Over the past, the EM method has adopted a layered-earth model as the standard of interpretation. Only recently have there been commercial, active EM sensors that can measure continuous wideband responses. While the EM method is popular for locating isolated conductivity anomalies (as in mineral and environmental surveys), there is a strong possibility that it can be expanded to a general stratigraphic mapping tool using broadband data. A theoretical analogue would be a frequency-domain reflection seismic section (of course, there is no such thing in practice!) that can be converted to a familiar time-domain section through an inverse Fourier transform. A 1-D earth having a continuous conductivity in depth, which may be derived from broadband EM data, would be the first step necessary to qualify the EM method as a general geologic mapping tool. Depth migration, similar in concept to that of the CDP seismics, can follow to enhance lateral resolution.

The possibilities of such a breakthrough are still in the exploratory stages, considering the complex physical principles and the geological materials involved. We hope that the formulation presented here encourages further stud-

ies in the geophysical community. We believe that the broadband EM method, when it can deal with the conductivity continuum, will be one of the most efficient geophysical methods for mapping complex subsurface structures encountered in geological, environmental, and geotechnical investigations.

#### Acknowledgment

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